Linear differential equation

The change in net earnings G as advertising expenditure x changes is given by

$$\frac{\partial G}{\partial x} = -b(G + x)$$

where b is a constant. Find G as a function of x if $G = G_1$ when $x = 0$.

Solution

We rewrite

$$G'_x + bG = -bx$$

It is a linear differential equation. We use the substitution:

$$G = uv$$

Then

$$G'_x = u'_x v + v'_x u$$

Replacing:

$$u'_x v + v'_x u + buv = -bx$$

$$v(u'_x + bu) + v'_x u = -bx$$

We solve the following system of equations:

$$u'_x + bu = 0$$

$$v'_x u = -bx$$

From the first equation.

$$\frac{du}{dx} = -bu$$

$$du/u = -bdx$$

Integrating both sides:

$$\ln(u) = -bx + K$$

$$u = e^{-bx+K} = Ce^{-bx}$$

Replacing in the other equation:

$$v'_x Ce^{-bx} = -bx$$

$$\frac{dv}{dx} Ce^{-bx} = -bx$$

$$Cdv = -bxe^{bx} dx$$

We integrate both sides. On the right side:

$$\int -bxe^{bx}\, dx = -\int bxe^{bx}\, dx$$

We use substitution:

$$w = bx$$

Then:

$$dw = bdx$$

$$dw/b = dx$$

$$-\int we^w \left(\frac{dw}{b}\right) = -\frac{1}{b}\int we^w\, dw$$

Integration by parts:

$$-\frac{1}{b}\left(\int we^w\, dw\right) = -\frac{1}{b}(we^w - \int e^w\, dw) = -\frac{we^w}{b} + \frac{e^w}{b} + T$$

Replacing back $w = bx$

$$-\frac{we^w}{b} + \frac{e^w}{b} + T = -xe^{bx} + \frac{e^{bx}}{b} + T$$

Then the result is:

$$Cv = -xe^{bx} + \frac{e^{bx}}{b} + T$$

Thus we solve for v:

$$v = -\frac{xe^{bx}}{C} + \frac{e^{bx}}{Cb} + J$$

The result is then:

$$G = uv = Ce^{-bx}\left(-\frac{xe^{bx}}{C} + \frac{e^{bx}}{Cb} + J\right) = -x + \frac{1}{b} + Ne^{-bx}$$

Where N is a constant. Using the point:

$$G(0) = G_1$$

$$G(0) = \frac{1}{b} + N = G_1$$

Solving for N:

$$N = G_1 - \frac{1}{b}$$

$$G = -x + \frac{1}{b} + \left(G_1 - \frac{1}{b}\right)e^{-bx}$$